Letting $\tau \to \infty$, we get

$$\bar{\Phi}_0 = {}^1/_2 (\beta\beta_1)^{-1} (2u_0^2 + \beta a_\eta^2), \quad \beta > 0, \ \beta_1 > 0$$

Thus, the system is unstable if $b < a_\zeta^2/2$. The variance of the velocity at $\tau \to \infty$, $u_0 = 0$, also depends on a_ζ. Consequently, the nature of the scattering of dissipative forces has a considerable effect on the dynamics of the system.

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THE SELFCONSISTENT PROBLEM OF THE VIBRATIONS OF AN INFINITE STRING LOADED WITH A MOVING POINT MASS*

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The problem of the vibrations of a homogeneous infinite string loaded with a point mass, moving in accordance with an unknown law of motion, is considered. This is one of the simplest model selfconsistent problems (SPs) in the dynamics of one-dimensional distributed loaded Lagrangian systems /1/. A mathematical formulation of the problem is given and the conditions for the existence and uniqueness of a global solution are established. An analytical method, which in many cases produces an exact solution, is presented. As an illustration, the displacement of a point mass along a vibrating string, set in motion by an impulse communicated to the mass, is considered. Certain effects related to the reverse action of the radiation of the moving point mass (braking by the radiation) are explained.

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Selfconsistent problems (SPs) arise in cases in which lumped factors (point loads or masses, lumped electric charges, etc.) in a distributed loaded Lagrangian system are not only subject to the effect of the system but exert their own influence on the latter. Allowance for this interaction in the formulation of SPs makes it possible to explain certain effects which are usually ignored /2/. SPs reduce to boundary-value problems in regions with moving and not previously known boundaries (the free boundary-value problem /3/). despite the natural origin of SPs, there is no general proof that they are well-posed; in fact, analytical methods for solving them have not been developed.

1. We consider a mechanical system consisting of a homogeneous, infinite string of linear density ρ, under tension μ, and a bead of mass m threaded on the string. We shall assume that in its rest state the string is straight and coincides with the x axis; it is capable of performing small transverse vibrations in the xu plane, while the bead can move without friction along the string. Given the initial configuration and the initial velocity of the system, it is required to determine its motion during the time interval $0 \leqslant t < +\infty$.

Thus, the vibrations of the string are described by a certain function $u(x, t)$, $-\infty < x < +\infty$, $t \geqslant 0$, and the motion of the bead is described by a vector-valued function with components $\chi(t), \eta(t), t \geqslant 0$, where $u(\chi(t), t) = \eta(t)$, $t \geqslant 0$. It is assumed that $\chi(t) \in C^2(t \geqslant 0)$, $\eta(t) \in C^3(t \geqslant 0)$, and moreover $|\chi^{\cdot}(t)| < a = \sqrt{\mu/\rho}$, $t \geqslant 0$, so that the curve γ: $x = \chi(t)$, $t \geqslant 0$, is a timelike curve /4/ in that xt plane relative to the homogeneous wave equation $u_{tt} = a^2 u_{xx}$ lying (with the exception of its enpoint $(x_0, 0)$, $x_0 = \chi(0)$) inside the future light cone $\Gamma = \{(x, t): |x - x_0| > at, \quad t > 0\}$. As to the function $u(x, t)$, it is assumed that $u(x, t) \in C(-\infty < x < +\infty, t \geqslant 0)$, and in fact $u(x, t) \in C^2(-\infty < x < +\infty, t \geqslant 0)$, except for the points on the curve γ and on the characteristics passing through $(x_0, 0)$, where the partial derivatives of $u(x, t)$ may have discontinuities of the first kind. Under these conditions we shall say that $\chi(t), \eta(t), u(x, t)$ are admissible functions.

Relying on previously obtained results /1/, we arrive at the following SP for the triple of admissible functions $\chi(t), \eta(t)$ and $u(x, t)$ describing the motion of the mechanical system in question:

$$u_{tt} = a^2 u_{xx} \tag{1.1}$$

$$\begin{gathered}
u(x, 0) = \varphi(x) = \begin{cases} \varphi_1(x), & -\infty < x \leqslant x_0 \\ \varphi_2(x), & x_0 \leqslant x < +\infty \end{cases} \\
u_t(x, 0) = \psi(x) = \begin{cases} \psi_1(x), & -\infty < x \leqslant x_0 \\ \psi_2(x), & x_0 \leqslant x < +\infty \end{cases}
\end{gathered} \tag{1.2}$$

$$m\eta^{\cdot\cdot} = \rho(a^2 - \chi^{\cdot 2})[u_x] \tag{1.3}$$

$$2m\chi^{\cdot\cdot} = -\rho(a^2 - \chi^{\cdot 2})[u_x^2] \tag{1.4}$$

$$u(\chi(t), t) = \eta(t) \tag{1.5}$$

$$\chi(0) = x_0, \quad \chi^{\cdot}(0) = p_0, \quad \eta(0) = u_0, \quad \eta^{\cdot}(0) = q_0 \tag{1.6}$$

where the brackets denote the jump of the function in question across the curve γ in the direction of the positive x axis, evaluated at the point $(\chi(t), t)$. The initial data of the problem are assumed to satisfy the following conditions:

$$|p_0| < a, \quad \varphi \in C(-\infty < x < +\infty), \quad \varphi(x_0) = u_0$$
$$\varphi_1 \in C^2(x \leqslant x_0), \quad \varphi_2 \in C^2(x \geqslant x_0), \quad \psi_1 \in C^1(x \leqslant x_0), \quad \psi_2 \in C^1(x \geqslant x_0)$$

2. We will find the conditions which guarantee the existence and uniqueness of an admissible global solution of the SP (1.1)-(1.6) and describe procedure for determining it.

Let us assume that some admissible global solution $\chi(t), \eta(t), u(x, t)$ of the SP (1.1)-(1.6) exists. Then the function $u(x, t)$ is found among the admissible global solutions of the initial-value problem (1.1)-(1.3) for the one-dimensional wave equation (augmented by a jump condition) which satisfy conditions (1.4) and (1.5). Let us assume that an admissible global solution $u(x, t)$ of problem (1.1)-(1.3) exists. Extend $u(x, t)$ to negative values $t < 0$ by defining it there as zero, i.e., setting $U(x, t) = \theta(t) u(x, t)$, where $\theta(t)$ is the Heaviside unit function and go over to the appropriate generalized function. Using the relationship between the generalized and classical derivatives of $U(x, t)$, with due allowance for the initial conditions (1.2) and jump condition (1.3), we infer that an admissible global solution $u(x, t)$ of problem (1.1)-(1.3), extended as zero to $t < 0$, may be found among those generalized solutions of the one-dimensional wave equation

$$U_{tt} = a^2 U_{xx} + G(x, t)$$
$$G(x, t) = -m\rho^{-1}\eta^{\cdot\cdot}(t)\,\delta(x - \chi(t)) + \varphi(x)\,\delta^{\cdot}(t) + \psi(x)\,\delta(t)$$

which are admissible functions for $t \geqslant 0$. As is well-known /4/, a generalized solution $U(x, t)$ of the one-dimensional wave equations exists, is unique and can be expressed as a wave potential. We have

$$U(x, t) = E(x, t) * G(x, t), \quad E(x, t) = (2a)^{-1}\,\theta(t)\,\theta(at - |x|)$$

where $E(x, t)$ is a fundamental solution of the one-dimensional wave operator and the asterisk denotes convolution of functions of x, t. For $t \geqslant 0$ the solution takes the following form:

$$u(x, t) = \begin{cases} \Phi_1(x, t), & -\infty < x \leqslant -at + x_0 \\ \Psi^+(x, t) & -at + x_0 \leqslant x \leqslant \chi(t) \\ \Psi^-(x, t) & \chi(t) \leqslant x \leqslant at + x_0 \\ \Phi_2(x, t), & at + x_0 \leqslant x < +\infty \end{cases} \tag{2.1}$$

$$\Phi_k(x, t) = \frac{1}{2}\varphi_k(x - at) + \frac{1}{2}\varphi_k(x + at) + \frac{1}{2a}\int_{x-at}^{x+at}\psi_k(\tau)\,d\tau, \quad k = 1, 2$$

$$\Psi^{\pm}(x, t) = sq_0 - s\eta'\left(\beta^{\pm}\left(t \pm \frac{x}{a}\right)\right) + F_1\left(t - \frac{x}{a}\right) + F_2\left(t + \frac{x}{a}\right), \quad s = \frac{m}{2a\rho}$$

$$F_1(\xi) = \frac{1}{2}\varphi_1(-a\xi) + \frac{1}{2a}\int_{-a\xi}^{x_0}\psi_1(\tau)\,d\tau, \quad \xi \in [-x_0/a, +\infty)$$

$$F_2(\xi) = \frac{1}{2}\varphi_2(a\xi) + \frac{1}{2a}\int_{x_0}^{a\xi}\psi_2(\tau)\,d\tau, \quad \xi \in [x_0/a, +\infty)$$

where $\beta^+(\xi)$, $\xi \in [x_0/a, +\infty)$ and $\beta^-(\xi)$, $\xi \in [-x_0/a, +\infty)$ are the inverses of the functions $\alpha^+(t) = t + \chi(t)/a$, $\alpha^-(t) = t - \chi(t)/a$, $t \in [0, +\infty)$, respectively.

Clearly, $u(x, t)$ is an admissible function. Thus an admissible global solution of problem (1.1)-(1.3) exists, is unique and is given by formula (2.1).

Thus, if the triple of admissible functions $\chi(t)$, $\eta(t)$, $u(x, t)$ is a global solution of the SP (1.1)-(1.6), then $u(x, t)$ is given by formula (2.1) and satisfies conditions (1.4), (1.5). Hence it follows that the pair of functions $\chi(t)$, $\eta(t)$ which are the other elements of the triple may be found among the admissible global solutions of the following initial-value problem for a canonical system of ordinary second-order differential equations:

$$\chi^{\cdot\cdot} = -[s(a^2 - \chi^{\cdot 2})]^{-1}\,[\eta^{\cdot} - f^+(t, \chi, \chi^{\cdot})]\,[\eta^{\cdot}\chi^{\cdot} + af^-(t, \chi, \chi^{\cdot})] \tag{2.2}$$
$$\eta^{\cdot\cdot} = -s^{-1}\,[\eta^{\cdot} - f^+(t, \chi, \chi^{\cdot})]$$
$$\chi(0) = x_0, \quad \chi^{\cdot}(0) = p_0, \quad \eta(0) = u_0, \quad \eta^{\cdot}(0) = q_0$$
$$(f^{\pm}(t, \chi, \chi^{\cdot}) = (1 - \chi^{\cdot}/a)\,F_1'(t - \chi/a) \pm (1 + \chi^{\cdot}/a)\,F_2'(t + \chi/a))$$

Conversely, if the pair of functions $\chi(t)$, $\eta(t)$ is an admissible global solution of problem (2.2), then the triple $\chi(t)$, $\eta(t)$, $u(x, t)$, where $u(x, t)$ is given by (2.1), is an admissible global solution of the SP (1.1)-(1.6), which can be verified directly.

To obtain conditions guaranteeing the existence and uniqueness of an admissible global solution of problem (2.2), one uses some suitable a priori estimate of its admissible local solutions. Assume that the derivatives F_1' and F_2' are square-integrable over the intervals $[-x_0/a, +\infty)$, $[x_0/a, +\infty)$, respectively. The law of conservation of energy for our mechanical system, formulated in terms of formulae (2.1), implies that if there exists an admissible local solution $\chi(t)$, $\eta(t)$ of problem (2.2) which is defined in some interval $[0, T]$, then it satisfies the "energy inequality"

$$\chi^{\cdot 2}(t) + \eta^{\cdot 2}(t) \leqslant V_0^2, \quad t \in [0, T] \tag{2.3}$$

$$V_0 = \left(p_0^2 + q_0^2 + \frac{1}{s}\int_{-x_0/a}^{\infty} F_1'^2(\xi)\,d\xi + \frac{1}{s}\int_{x_0/a}^{\infty} F_2'^2(\xi)\,d\xi\right)^{1/2} = \text{const} \geqslant 0 \tag{2.4}$$

Inequality (2.3) yields an a priori estimate for admissible local solutions $\chi(t)$, $\eta(t)$ of problem (2.2) which are defined for $0 \leqslant t \leqslant T$, namely:

$$|\chi^{\cdot}(t)| \leqslant V_0, \quad |\chi(t) - x_0| \leqslant V_0 t, \quad |\eta^{\cdot}(t)| \leqslant V_0, \quad |\eta(t) - u_0| \leqslant V_0 t \tag{2.5}$$

The quantity V_0 of formula (2.4) will be called the defining parameter of the SP (1.1)-(1.6).

Lemma. If the defining parameter V_0 of the SP (1.1)-(1.6) satisfies the condition

$$V_0 < a \tag{2.6}$$

then problem (2.2) corresponding to this SP has an admissible global solution, which is moreover unique.

Indeed, let T be an arbitrary positive number. Define $a - V_0 = 2d$ and, in the extended phase space $(t, \chi, \chi^{\cdot}, \eta, \eta^{\cdot})$, consider a parallelepiped Π: $0 \leqslant t \leqslant T$, $|\chi - x_0| \leqslant (V_0 + d)\, T$, $|\chi^{\cdot}| \leqslant V_0 + d$, $|\eta - u_0| \leqslant (V_0 + d)\, T$, $|\eta^{\cdot}| \leqslant V_0 + d$, in which the conditions of the local existence and uniqueness theorem for the initial-value problem are satisfied. By the continuation theorem /5/, the local solution, which is obviously admissible, may be continued up to the boundary Π. It follows from the a priori estimate (2.5) that the solution can reach only the boundary $t = T$. Thus an admissible local solution can be continued to the right up to any $t = T > 0$, and hence it is unrestricted in that direction. Since the continuation is unique, our assertion is proved.

Let $\chi(t)$, $\eta(t)$ be an admissible global solution of problem (2.2). Then $\eta(t)$ is an admissible global solution of a second-order ordinary differential equation (the second equation of (2.2) on the assumption that $\chi(t)$ is a given function) satisfying the initial conditions $\eta(0) = u_0$, $\eta^{\cdot}(0) = q_0$. This equation is readily integrated in quadratures, since it has a first integral which is a first-order linear differential equation. We have

$$\eta(t) = sq_0 + e^{-t/s}\left\{u_0 - sq_0 + \frac{1}{s}\int_0^t \left[F_1\left(\tau - \frac{\chi(\tau)}{a}\right) + F_2\left(\tau - \frac{\chi(\tau)}{a}\right)\right] e^{\tau/s} d\tau\right\} \tag{2.7}$$

Consequently, $\chi(t)$ is an admissible global solution of the following initial-value problem for an integrodifferential equation

$$\chi^{\cdot\cdot} = \eta^{\cdot\cdot}(t)\,(a^2 - \chi^{\cdot 2})^{-1}\,[\eta^{\cdot}(t)\,\chi^{\cdot} + af^{-}(t, \chi, \chi^{\cdot})] \tag{2.8}$$
$$\chi(0) = x_0, \quad \chi^{\cdot}(0) = p_0$$

where $\eta(t)$ is given by (2.7). If condition (2.6) is satisfied, this problem has an admissible global solution, which is moreover unique.

We have thus established the following assertion: If the defining parameter V_0 of the SP (1.1)-(1.6) satisfies condition (2.6), then a unique admissible global solution; $\chi(t)$, $\eta(t)$, $u(x, t)$ of the problem exists; the function $\chi(t)$ is an admissible global solution of problem (2.8), and $\eta(t)$ is given by formula (2.7) and $u(x, t)$ by formula (2.1).

3. *Example.* Let us consider a bead of mass m moving along an infinite vibrating homogeneous string, set in motion at time $t = 0$ by a momentum $m\mathrm{v}_0$ with components mp_0, mq_0 communicated to the bead, on the assumption that $|\mathrm{v}_0| < a$.

To determine the vector-valued function with components $\chi(t)$, $\eta(t)$ and the function $u(x, t)$, describing the motion of the bead and the string, respectively, we have a SP with initial data

$$u(x, 0) = 0, \quad u_t(x, 0) = 0, \quad \chi(0) = x_0, \quad \chi^{\cdot}(0) = p_0, \quad \eta(0) = 0, \quad \eta^{\cdot}(0) = q_0$$

characterized by a defining parameter $V_0 = |\mathrm{v}_0| < a$.

The function $\eta(t)$ is given by

$$\eta(t) = sq_0(1 - e^{-t/s}), \quad s = \frac{m}{2a\rho}, \quad t \geqslant 0 \tag{3.1}$$

and $\chi(t)$ is an admissible global solution of the following Cauchy problem for a second-order ordinary differential equation:

$$s(a^2 - \chi^{\cdot 2})\,\chi^{\cdot\cdot} = -q_0^2 e^{-2t/s}\chi^{\cdot}; \quad \chi(0) = x_0, \quad \chi^{\cdot}(0) = p_0 \tag{3.2}$$

If $p_0 = 0$, then obviously $\chi(t) = x_0$, $t \geqslant 0$; if $q_0 = 0$, then $\chi(t) = p_0 t + x_0$, $t \geqslant 0$ (in which case $\eta(t) = 0$, $t \geqslant 0$).

Now let $p_0 \neq 0$, $q_0 \neq 0$. If $p_0 > 0$, then $\chi^{\cdot}(t) > 0$, $\chi^{\cdot\cdot}(t) < 0$, $t \geqslant 0$, so that $\chi^{\cdot}(t)$ decreases monotonically from p_0 (at $t = 0$) to p_+ (at $t = +\infty$), where p_+ is the root of the equation

$$\Delta(p) \equiv p_0^2 - a^2 \ln p_0^2 + q_0^2 - p^2 + a^2 \ln p^2 = 0 \tag{3.3}$$

in the interval $0 < p <$ [illegible]. If $p_0 < 0$, then $\chi^{\cdot}(t) < 0$, $\chi^{\cdot\cdot}(t) > 0$, $t \geqslant 0$, so that $\chi^{\cdot}(t)$ increases monotonically from p_0 (at $t = 0$) to p_- (at $t = +\infty$), where p_- is the root of Eq.(3.3) in the interval $p_0 < p < 0$. The function $x = \chi(t)$ is found in parametric form (with parameter p):

$$t = \frac{s}{2} \ln \frac{q_0^2}{\Delta(p)}, \quad x = x_0 - s \int_{p_0}^{p} \frac{a^2 - \xi^2}{\Delta(\xi)} d\xi \tag{3.4}$$

where $p_0 \geqslant p > p_+$ (if $p_0 > 0$) and $p_0 \leqslant p < p_-$ (if $p_0 < 0$).

The function $u(x, t)$, $t \geqslant 0$, is given by

$$u(x, t) = \begin{cases} 0, & -\infty < x \leqslant -at + x_0 \\ \eta(\beta^+(t + x/a)), & -at + x_0 \leqslant x \leqslant \chi(t) \\ \eta(\beta^-(t - x/a)), & \chi(t) \leqslant x \leqslant at + x_0 \\ 0, & at + x_0 \leqslant x < +\infty \end{cases} \tag{3.5}$$

where η is given by formula (3.1) and β^+, β^- are the inverses of the functions $\alpha^+(t) = t + \chi(t)/a$, $\alpha^-(t) = t - \chi(t)/a$, $t \geqslant 0$, respectively.

As follows from our analysis, in accelerated motion the bead will continuously lose energy; in uniform motion, no energy is lost. We note that similar effects due to the radiating reaction of a moving lumped factor (braking by the radiation), are observed in other interactive processes (e.g., interaction of a charged particle with its own electromagnetic field /6/).

4. *Remarks*. 1. A similar method may be used to solve SPs concerning the vibrations of an infinite string loaded with several moving point masses which do not collide while in motion. Applying the method of reflections, one can also obtain solutions to SPs concerning the vibrations of semi-infinite and even bounded strings loaded with one or several moving point masses which do not collide with one another or with the end of the string while in motion.

2. The SP (1.1)-(1.6) is essentially a mixed Cauchy problem with free boundary /7/ for the one-dimensional wave equation. The method of solution proposed in this paper, which is based on the idea of isolating different motions in the system, may be extended to SPs arising from other equations of hyperbolic type with constant coefficients, since for such equations a solution of the generalized Cauchy problem exists, is unique and can be represented as a convolution of a fundamental solution (with support contained in a certain convex cone) and a source /8/.

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